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Analysis of Exponential and Hyperbolic Growth Models of Population Dynamics Applied to Indian Population Growth

Muthukamalam MS*

Department of Mathematics, Nadar Saraswathi College of Arts & Science, Tamil Nadu, India

Research Article

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***For Correspondence**

M. S. Muthukamalam, Department of Mathematics, Nadar Saraswathi College of Arts & Science, Tamil Nadu, India

E-mail: muthukamalamms@gmail.com

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ABSTRACT

India is a country that occupies the greater part of South Asia. India lies on the Indian Plate, the northern part of the Indo-Australian Plate, whose continental crust forms the Indian subcontinent. It has an area of 3,287,263 square kilometers. With roughly one-sixth of the world's total population, India is the second most populous country, after China. Using only two data points, an exponential growth population model applied to Indian population is developed and used both to project future population of India and compare to past population data of India. A hyperbolic growth model for Indian population is then developed, and its fits to prior population data are compared with the exponential model. Expressions for doubling times are derived from both models and compared to real data.

INTRODUCTION

Mathematical modeling is a mathematical tool in treating the use of mathematics in explaining phenomena arising in the environment. This tool is being used widely in the field of applied mathematics and engineering. This is the process of illustrating the reality in terms of the language of mathematics commonly by the aid of mathematical and computational techniques. Mathematical models made use of the concepts of higher approach in mathematics commonly using differential equations and numerical methods and analysis. Population model is a mathematical model applied on population dynamics. Mathematical modeling can be used for a number of different reasons. The objectives include:

- developing scientific understanding;
- test the effect of change in a system; and
- aid decision making including tactical decisions by managers and strategic decisions by planners [1].

Models and methods have been used in producing forecasts of population growth. The work is intended to emphasize the reliability bounds of the model forecasts [2] . A standard part of the calculus curriculum is learning exponential growth models. This paper extends the standard modeling by showing that simple exponential models, relying on two points to fit parameters do not do a good job in modeling population data of the distant past. Moreover, they provide a constant doubling time. Therefore, hyperbolic modeling is introduced and it is demonstrated that with only two population data points, amazing information can be obtained, such as reasonably accurate doubling times that are a function of t [3] .

OBJECTIVES

- To develop a simple exponential model of the population growth by understanding its assumptions. (Let us consider the initial values from 1950 and 2020)
- This will give confidence in using the model to forecast the future population growth. (For Example, when will the Indian population reach 5 billion or 10 billion what will be the population of India in 2050)
- Develop Hyperbolic Growth Model, as an illustration uses a relative growth rate which is not a constant, but a function of population. The corresponding differential equation is developed and solved.
- To compare both Exponential Growth Model and Hyperbolic Growth Model to recent population data it would be easy to

visualize that there exists an excellent fit between actual and predicted population values.

- Develop expressions for doubling times using both models, and learn the difference between a constant doubling time (as in the exponential growth model) and a time-dependent doubling time (as in hyperbolic growth model).
- This project will appreciate that different models can be used to describe the same data each with its own strengths and weaknesses.

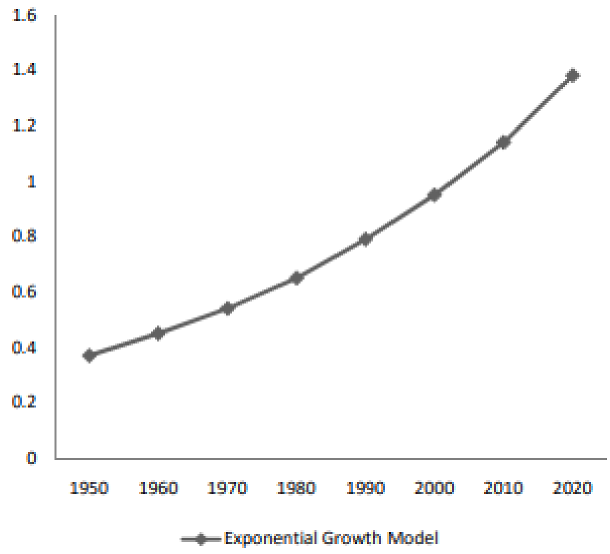

Figure 1: Exponential Growth Model

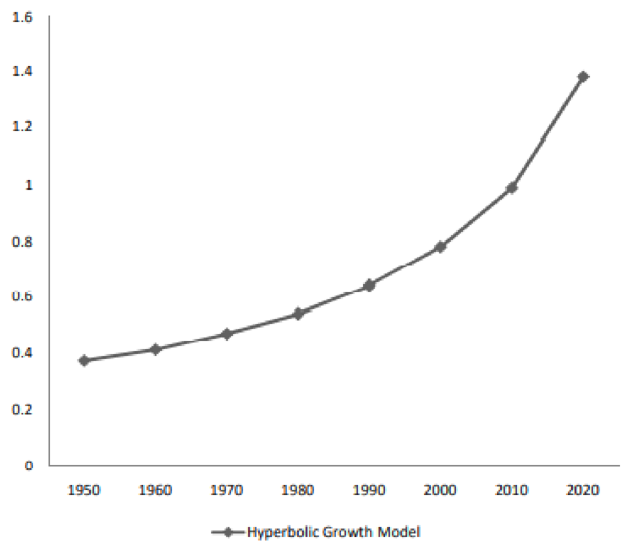

Figure 2: Hyperbolic Growth Model

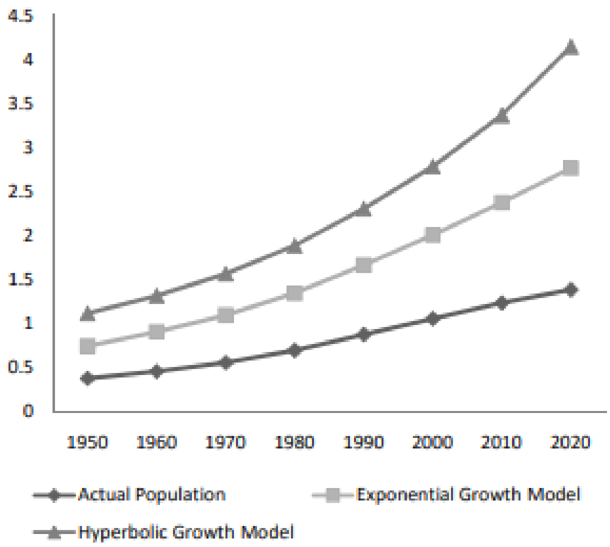

Figure 3: Comparison between the exponential and hyperbolic growth model with real data

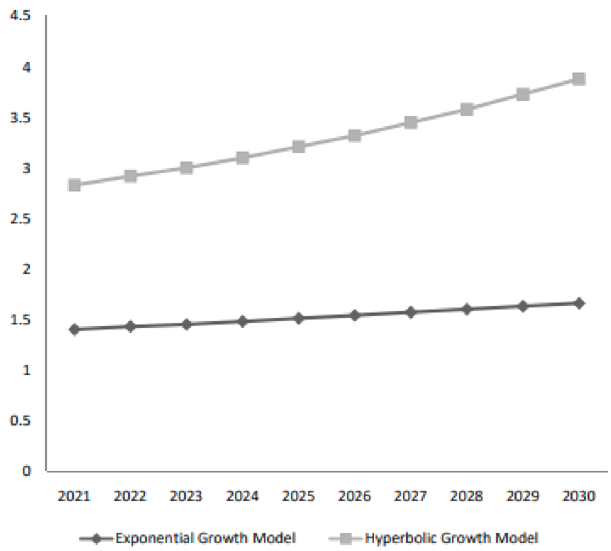

Figure 4: Project population from 2021 to 2030

ANALYSIS OF THE MODELS

Exponential growth model

According to this model, "the time rate of change of population is proportional to the present population", therefore, mathematically,

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = kP$$

By Separation of variables,

$$\frac{dP}{P} = kdt$$

Integrating the above equation, we get,

$$\int \frac{1}{P} dP = k \int dt$$

$$\log P = kt + c$$

Taking exponential function on both sides, we get,

$$P(t) = e^{kt+c}$$

$$= e^{c}e^{kt}$$

$$P(t) = \mathrm{C}e^{kt}$$

Using only two data from sample points, here let us take 1950 and 2020, where the Indian population was 0.376325 and 1.380004 billion respectively; we can determine P0 and k as follows. We set 1950 as t = 0 time point by recasting the model as follows:

$$P(t) = \mathrm{C}e^{k(t-1950)}$$

Since in 1950, the population was 0.376325 billion, this makes,

$$C = P_0 = 0.376325$$

So, we can write our model as:

$$P(t) = \mathrm{P}_0\ e^{k(t-1950)}$$

Now, using the population data from 2020, we can solve for k as follows:

$$k = \frac{\ln\left(\dfrac{P(2020)}{P(1950)}\right)}{70}$$

$$= \frac{\ln\left(\dfrac{1.380004}{0.376325}\right)}{70}$$

$$= \frac{\ln(3.667053743)}{70}$$

$$= \frac{1.299388545}{70}$$

k = 0.018562693

Thus, the exponential model can be written as:

$$P(t) = 0.376325e^{0.018562(t-1950)}$$

Example, Thus using our model, we can plug in t = 1985 to estimate the population in that year in billions,

$$P(1985) = 0.376325e^{0.018562(1985-1950)}$$

$$P(1985) = 0.376325e^{0.018562(35)}$$

$$P(1985) = 0.376325(1.914955252)$$

$$P(1985) = 0.720645535$$

Hyperbolic growth model

With the exponential growth model, the differential equation is,

$$\frac{dP}{dt} = kP$$

Where we assumed that k, the relative growth rate, was a constant. Now, we would like to explore altering this assumption, where in general k becomes a function of P, i.e.,

$$\frac{dP}{dt} = k(P)P$$

Probably the simplest such function, used in what are called hyperbolic growth models, is to assume that k is proportional to P. This model makes some sense, in that as population grows, this may indicate favorable economic conditions or social conditions which encourage people to have more children, and so the rate of population increase actually grows with the population. To maintain the dimensions in the equation, we can say that,

$$\frac{dP}{dt} = \frac{kP}{P_0}$$

where, on the right, k is now a constant multiplied by P, which varies with time, and P0 is a constant. Now, our differential equation becomes:

$$\frac{dP}{dt} = \mathrm{k(P)P} = \frac{kP}{P_0}P$$

$$= \frac{kP^2}{P_0}$$

Now, we need to solve the differential equation,

$$\frac{dP}{dt} = \frac{kP^2}{P_0}$$

This can be done by separation of variables, so that we have,

$$\frac{dP}{P^2} = \left(\frac{k}{P_0}\right)dt$$

Now, we can integrate both sides, such that,

$$\int \frac{dP}{P^2} = \int \left(\frac{k}{P_0}\right)dt$$

This gives us that,

$$-\frac{1}{P} = \left(\frac{k}{P_0}\right)t + \mathrm{C}$$

Now, if we say that the population is P0 at time t = 0, then we see that,

$$\mathrm{C} = -\frac{1}{P_0}$$

$$-\frac{1}{P} = \frac{k}{P_0}t - \frac{1}{P_0}$$

$$-\frac{1}{P} = \frac{k}{P_0}t - \frac{1}{P_0}$$

$$-P = \frac{P_0}{kt-1}$$

$$P = -\frac{P_0}{kt-1}$$

So, we have as our hyperbolic growth model,

$$P = \frac{P_0}{1-kt}$$

Once again, we can use data from 1950 and 2020 to get our constants. We let 1950 be our zero time point, and so P0 = P (1950) = 0.376325, and use this to find k, by manipulating the equation above:

$$P(2020) = \frac{P_0}{1-k(2020-1950)}$$

$$= \frac{0.376325}{1-k(70)}$$

$$k = \frac{1-\dfrac{0.376325}{P(2020)}}{70}$$

$$= \frac{1-\dfrac{0.376325}{1.380004}}{70}$$

$$= \frac{1-0.272698484}{70}$$

$$= \frac{0.727301516}{70}$$

$$k = 0.01039002$$

$$P(t) = \frac{0.376325}{1 - 0.01039002(t - 1950)}$$

Example, Thus using our model, we can plug in t = 1985 to estimate the population in that year in billions,

$$P(1985) = \frac{0.376325}{1 - 0.01039002(1985 - 1950)}$$

$$= \frac{0.376325}{1 - 0.01039002(35)}$$

$$= \frac{0.376325}{1 - 0.3636507}$$

$$P(1985) = \frac{0.376325}{0.6363493}$$

$$P(1985) = 0.59138118$$

Comparing the exponential and hyperbolic growth model to real data

Table 1: Comparing the exponential and hyperbolic growth model to real data

Year	Actual Population (in billions) [4]	Exponential Growth Model (in billions)	Hyperbolic Growth Model (in billions)
1950	0.376325	0.376325	0.376325
1951	0.382377	0.383375	0.380276
1952	0.388799	0.390558	0.384310
1953	0.395544	0.397876	0.388432
1954	0.402579	0.405330	0.392643
1955	0.409881	0.412925	0.396946
1956	0.417443	0.420661	0.401345
1957	0.425271	0.428543	0.405841
1958	0.433381	0.436572	0.410440
1959	0.441799	0.444752	0.415145
1960	0.450548	0.453085	0.419959
1961	0.459642	0.461574	0.429929
1962	0.469077	0.470222	0.429929
1963	0.478826	0.479032	0.435093
1964	0.488848	0.488008	0.440383
1965	0.499123	0.497150	0.445803
1966	0.509632	0.506466	0.451359
1967	0.520401	0.515954	0.457055
1968	0.531514	0.525621	0.462896
1969	0.543084	0.535470	0.468888
1970	0.555190	0.545502	0.475038
1971	0.567868	0.555722	0.481351
1972	0.581087	0.566134	0.487834
1973	0.594770	0.576742	0.494495
1974	0.608803	0.587547	0.501339
1975	0.623103	0.598556	0.508375
1976	0.637630	0.609771	0.515613
1977	0.652409	0.621195	0.523059
1978	0.667500	0.632834	0.530723
1979	0.682995	0.644691	0.538616
1980	0.698953	0.656770	0.546746
1981	0.715385	0.669075	0.555126

1982	0.732240	0.681611	0.563766
1983	0.749429	0.694381	0.572680
1984	0.766833	0.707392	0.581881
1985	0.784360	0.720646	0.591381
1986	0.801975	0.734148	0.601134
1987	0.819682	0.747903	0.611345
1988	0.837469	0.761915	0.621841
1989	0.855335	0.776191	0.632703
1990	0.873278	0.790733	0.643951
1991	0.891273	0.805549	0.655608
1992	0.909307	0.820641	0.667694
1993	0.927404	0.836017	0.680233
1994	0.945602	0.851680	0.693253
1995	0.963923	0.867638	0.706781
1996	0.982365	0.883893	0.720848
1997	1.000900	0.900454	0.735485
1998	1.019484	0.917326	0.750729
1999	1.038058	0.934512	0.766619
2000	1.056576	0.952022	0.783196
2001	1.075000	0.969859	0.800506
2002	1.093317	0.988030	0.818598
2003	1.111523	1.006541	0.837527
2004	1.129623	1.025400	0.857351
2005	1.147610	1.044612	0.878137
2006	1.165486	1.064184	0.899956
2007	1.183209	1.084122	0.922888
2008	1.200670	1.104435	0.947018
2009	1.217726	1.125127	0.972446
2010	1.234281	1.146208	0.999273
2011	1.250288	1.167683	1.027624
2012	1.265780	1.189562	1.057631
2013	1.280842	1.211849	1.089443
2014	1.295600	1.234554	1.123228
2015	1.310152	1.257685	1.159176
2016	1.324517	1.218249	1.197501
2017	1.338677	1.305255	1.238446
2018	1.352642	1.329710	1.282290
2019	1.366418	1.354623	1.329354
2020	1.380004	1.380004	1.380003
MAPE		5.0568959 %	16.482499 %
Interpretation		Highly Accurate	Good

Table 2: Projected Population

Year	Exponential Growth Model (in billions)	Hyperbolic Growth Model (in billions)
2021	1.405859	1.434665
2022	1.432200	1.493835
2023	1.459033	1.558097
2024	1.483700	1.628136
2025	1.514219	1.704768
2026	1.542589	1.788969
2027	1.571491	1.881921
2028	1.600935	1.985062
2029	1.630930	2.100163
2030	1.661487	2.022943

The results of the population based on exponential and hyperbolic growth models are given below;

Based on the results of the computed MAPE of the individual model, it is found that the exponential growth model has a MAPE of 5.056896 % which is highly accurate; the hyperbolic model has a MAPE of 16.482470 % with an interpretation of good. Comparing the models, it is found that the exponential growth model is more accurate than hyperbolic growth model. This is due to the fact that the lowest MAPE obtained implies that it is the most accurate prediction.

PROJECTED POPULATION

The projected population from 2021 to 2030 using exponential and hyperbolic growth models is given below;

DOUBLING TIMES

One very interesting issue about populations is how long it takes the population to double, known as the doubling time. We can use both of our models to predict the doubling times. For the exponential model, let us say we observe the population to be P (t1) at some point t1, and we want the time interval it takes for the population to double:

$$P(t_1) = \mathrm{C}e^{kt_1}$$

Then at time t2, the population has doubled and we have

$$P(t_2) = 2\mathrm{P}(t_1) = \mathrm{C}e^{kt_2}$$

Dividing the second equation by the first, we have,

$$2 = e^{k(t_2 - t_1)}$$

$$t_2 - t_1 = doublingtime = \frac{\ln 2}{k}$$

Thus, for the exponential model, the doubling time is a constant, and depends on the value of k. In our model,

$$k = 0.018562693 yr^{-1},$$

$$= \frac{0.69314718}{0.018562693}$$

$$= 37.34087401$$

And so the doubling time is about 37 years. This seems to fit pretty well with the modern data, where in 1966, the population was about 0.509632 billion people, and in 2003, the population was about 1.111523.billion people.

Using the hyperbolic growth model, we can also calculate an expression for the doubling time, starting with,

$$P(t_1) = \frac{P_0}{1 - kt_1} \; and \; P(t_2) = \frac{P_0}{1 - kt_2}$$

Therefore we see that for the hyperbolic growth model, the doubling time is not constant, but actually varies with time. Therefore, for the period ending in the year 2020, we see that,

$$= \frac{1 - 0.01039002(2020 - 1950)}{0.01039002}$$

$$= \frac{1 - 0.01039002(70)}{0.01039002}$$

$$= \frac{1 - 0.7273014}{0.01039002}$$

$$= \frac{0.2726986}{0.01039002}$$

$$(t_2 - t_1) = 26.2462054$$

And so the doubling time is about 26 years. Similar calculations for the period 1966, the population was 0.509632 which is quite close to the doubling time of the period 1992, the population was 0.909307.

CONCLUSION

Mathematical modeling of population growth provides an excellent tool both to predict and forecast future population growth, as well as to answer questions about the past. In this paper, we developed the Exponential and Hyperbolic Growth Models of the Indian Population growth and compared these models. One has to always test models against real data, and to carefully know the strengths and weaknesses of each model. Also, it is important to note that accurate predictions of population growth are much more complicated than the simple models presented above. Such predictions must take into account such factors as the age distribution of the population, birth rates, death rates, and scarcity of resources as a population grows. Based on the results, the MAPE of Exponential Growth Model is 5.0568959 % (Highly accurate) and the hyperbolic model is 16.4824699 % (Good). It is found that the exponential Growth Model is more accurate for its projection from 1950 to 2020. The projected population of the country using Exponential Growth Model is about 1.661487 billion and 2.229434 billion using Hyperbolic Growth Model by the year 2030.

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