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SEMI-INVARIANTS FORMS: ABSOLUTE FINITE REFLECTION CLASS

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Abstract: Let G be a finite group of complex $n \times n$ unitary matrices generated by reflections acting on C^n. Let R be the ring of invariant polynomials, and χ be a multiplicative character of G. Consider the R-module of χ-invariant deferential forms and the R-module of χ -invariants in the exterior algebra of derivations. We define a natural multiplication on these modules using ideas from arrangements of hyper planes. We show that this multiplication gives each module the structure of an exterior algebra. We also define a multi-arrangement associated to χ, and formulate the relationship between _-invariants and logarithmic forms. We introduce a new method of computing basic derivations and the generating _-invariants and give explicit constructions for the exceptional irreducible reflection groups.

Keywords- inverse problems, parameter estimation, systems of ordinary differential equations, short-cut methods, separable nonlinear least squares problems

INTRODUCTION

Mathematics is one of the most demanding and difficult subjects for a student to master [1] [2]. Mathematics is taught every year from the beginning of elementary education through post-secondary education and in graduate education. Basic mathematic skills are essential to everyday life. From shopping to traveling, “math problems” exist in every aspect of daily living. However, the emphasis placed on mathematics in education and the pervasive nature of mathematics in everyday life are not enough to motivate some students to learn, master, and retain its concepts.

Today mathematics is integrated into the educational systems of all developed countries. Most schools begin teaching addition and subtraction to students who are between the ages of five and seven years old. Students progress through mathematics courses focusing on different subdivisions of mathematics as they move through the grades in school. By the time the student graduates from high school or any secondary education institution, he will have been taught the major subdivisions of mathematics including algebra, geometry, and calculus [15].

A. Layperson's:

If you hold two mirrors together at a right angle and then place a key in front of the mirrors, you will see four keys - one real, three images. As you angle the mirrors closer together, you see more and more keys. Mathematicians like to place infinitely tall and wide mirrors in a space and examine how imagines bounce from mirror to mirror and end up in new locations. They fix some plane to act as a mirror, and then reflect vectors about that plane. This is accomplished on paper with a well-chosen matrix: when you apply the matrix to a vector, you get a new vector which is the image upon reflecting through the given plane. We are not really interested in the physics of reflecting light or mirrors. Rather, we would like to know how rearranging a space with reflections affects properties like length, volume, orientation (right changes to left!), etc. The distance between two points doesn't change after a series of reflections. What other functions on the space are unaffected? Mathematical objects are called invariant when they are unaffected and called semi-invariant when they are almost unaffected | they change by a constant. Semi-invariants are the subject of this paper [3]

B. Mathematician’s:

The present inquiry on semi-invariants arose from some questions about dynamical systems. In 1989, P. Doyle and C. McMullen [7] solved the fifth degree polynomial using a highly symmetrical dynamical system which preserved the Galois group A_5. In 1997, S. Crass and P. Doyle [4] tackled the sixth degree polynomial by again finding a dynamical system with special symmetry - this time A_6 symmetry. Each dynamical system was formed by iterating a map that was equivariant under the projective action of a reflection group. Such maps correspond naturally to semi-invariant differential forms. Because almost nothing was known about these forms, constructing the necessary dynamical systems was a difficult step in both cases.

Here, introduced a general theory of semi-invariants. Specifically, show that for any finite unitary reflection group G and multiplicative character χ of G, the module of χ-invariant differential forms has a natural multiplication which turns the module into an exterior algebra. This exterior algebra structure allows us to understand completely the forms that give rise to highly symmetrical dynamical systems, and gives us tools to compute these forms explicitly. Extend these results to vector fields (or derivations), observe some applications to logarithmic forms, and show new techniques for computing semi-invariants. The theory presented here builds on work by R. Stanley, who characterized the module of χ -invariant polynomials in 1977 [15]. It also builds on more recent work by P. Orlik, K. Saito, L. Solomon, H. Terao and others on invariant derivations [13].

SOME TERMINOLOGY

Here, define reflection groups, differential forms, derivation forms, some particular group actions, and semi-invariance [11]. These terminologies helpful in motivating this paper

A. Reflection Groups

Reflection groups are called real reflection groups, or Coxeter groups, when their matrices are real. Coxeter groups are generated by orthogonal reflections with determinant - 1. The symmetry group of a regular complex polytope, called a Shephard group, is also a reflection group. In 1954, G. Shephard and J. Todd [11] published a paper that proved many fundamental properties of reflection groups: Finite unitary reflection groups. They extended results about real reflection groups to general (complex) reflection groups, collected information about the groups, and proved important new properties. Every reflection group is either irreducible or the direct product of irreducible components, each of which is itself a reflection group.

Begin with a complex vector space V: = C^n. A unitary n × n matrix is a reflection if its fixed point set is a hyperplane of V, i.e., an (n-1) - dimensional space. A reflection matrix is characterized by the fact that n - 1 of its eigenvalues are 1 (corresponding to the fixed hyperplane) and the remaining eigenvalue is a non-trivial root of unity. If the non-trivial eigenvalue is a k-th root of unity, then we say that the reflection is k-fold. A reflection group is a finite group of matrices generated by reflections. They are often called finite reflection groups, finite pseudo-reflection groups, or U.G.G.R. (unitary groups generated by reflections).The second group of researchers adopting the external view espouses a more dynamic view of mathematics, but they focus on adjusting the curriculum to reflect this growth of the discipline and to see how students acquire knowledge of the related content and skills. The underlying focus is, however, still on student mastery of the curriculum or on the application of recent advances in technology or instructional technology to mathematics instruction.

B. Forms

Now some notation: Let S := C [x_1 ,…..,x_n] be the ring of polynomials on V and F := C (x_1,……,x_n) be the field of rational functions. Denote the module of differential p-forms on V by

$$\Omega^P := \bigoplus_{1\le i_1<..<i_{p\le n}} Sdx_{i_1}\wedge\ldots\wedge dx_{i_p}$$

$$\cong s\otimes\wedge^p V^*$$

Let $\partial x_i := \frac{\partial}{\partial x_i}$ and let

$$\Upsilon^p := \bigoplus_{1\le j_1<..<j_p\le n} s\partial x_{j_1}\wedge..\wedge\partial x_{j_p}$$

$$\cong s\otimes\wedge^p V^*$$

We agree that $\Omega^0 = \Upsilon^0 = S$.

It is helpful; to define a matrix od coefficients for forms. For β_1,….. β_n in γ^1 (or in Ω^1), write each β_j as

$$\beta_j = \sum_{i=1}^{n}\beta_j^{(i)}\partial x_i .$$

C. Group Actions

Let G be a reflection group. The group G not only acts on the space V, but also on S, Ω^p, and γ^p. The action on S is defined by gf := f∘g^{-1} for f ∈S, g ∈ G. The correspondence between x_i and $\partial x_i = \partial/\partial x_i$ extends anti-linearly to degree one polynomials by

$$\partial\left(\sum_{i=1}^{n} c_i x_i\right) := \sum_{i=1}^{n} C_i^1\partial x_i$$

Extend the action to

$$g\sum_{I\in I^P} h_1 dx_I := \sum_{IEIp} gh_I gdx_I .$$

INVARIANT POLYNOMIALS

The best known semi-invariants are those that are invariant: forms β that satisfy gβ = β for all g in G. A rich theory of invariants for reflection groups has developed around a powerful theorem by G. Shephard, J. Todd, and C. Chevalley [11] (V.5.3, Theorem 3) describing the set of invariant polynomials.

A. Invariants

Let R be the set of invariant polynomials. The celebrated theorem about invariant polynomials is [1]

Theorem 1 *There exist n homogeneous polynomials, $f_{1,\ldots\ldots,}f_n$, with R = C [f_1,…….,f_n]. The degrees of the f_i are uniquely determined.*

We call the polynomials in the above theorem basic invariants, and call R the ring of invariants. Basic invariants have been constructed for all 37 of the irreducible reflection groups. Notice that $(\Upsilon^p)^\chi$ and $(\Omega^p)^\chi$ are modules over R. In 1963, L. Solomon [14] showed that the R-module of invariant differential forms, Ω^G, has the beautiful structure of an exterior algebra:

Theorem 2 The module Ω^G is generated over R as an exterior algebra by the df_i, i.e., $(\Omega^p)^G$ is generated over R by the forms df_{i1}^ _ _ _ ^ df_{ip}, where 1 ≤ i_1 ,……, i_p ≤ n.

B. Application to Semi-invariants

Given any G-module N, we can define N^G := {n ∈ N : g(n) = n ∀g ∈ G}. We state a well-known proposition about G-modules; for proof, see Lemma 6.45 of [9].

Proposition *If M is a G-module of dimension r over C, then the R-module $(S\otimes M)^G$ is free of rank r (over R).*

Corollary *The R-modules $(\Omega^p)^\chi$ and $(\Upsilon^p)^\chi$ are both free of rank $\binom{n}{p}$.*

ARRANGEMENTS OF HYPERPLANE

Reflection groups are often studied using results from arrangements of hyperplane. One important result completely describes semi-invariant polynomials. Here, explained the connection between reflection groups and hyperplane arrangements, and give the fundamental result on semi-invariant polynomials. We will often follow notation from the wonderful text Arrangements of Hyperplane [9], which should be consulted as a general reference.

A hyperplane in W is a (n-1)-dimensional affine subspace of W. A hyperplane arrangement is a finite set of hyperplane. For each hyperplane H in a hyperplane arrangement A, let α_H be a linear polynomial on W whose kernel is H. We call

$$Q(A) = \prod_{H\in A} \alpha H$$

the defining polynomial of A. The polynomial Q(A) is uniquely defined up to a nonzero scalar multiple

A. Reflection Arrangements

We now consider the arrangement defined by our reflection group G. Each reflection in our group fixes a hyperplane in C n. Fix A as the collection of all such hyperplane. Notice that the group G permutes the hyperplane in A. For each $H \in A$, define $\alpha_H \in S$ by $\ker(\alpha_H) = H$. Then the polynomial

$$Q := Q(A) = \prod_{H\in A} \alpha H$$

defines the hyperplane arrangement A.

EXTERIOR ALGEBRA

A. □-Wedging

The first step in understanding the structure of Ω^χ is to define a multiplication. Observe that Ω^χ is not closed under the exterior product! Here unwinds the definitions of Q_χ and the group action in a helpful coordinate system. Recall that

$$Q_\chi = \prod_{H\in A} \alpha_H^{a_H(\chi)}$$

and that s_H is a reflection in G of maximal order that fixes H point wise. So Q_χ divides the exterior product of any two $\chi-$ invariant differential forms. if write

$$\mu = \sum_{I\in I^P} \mu_I dx_{I1} \wedge \ldots \wedge dx_{Ip},$$

and

$$\omega = \sum_{J\in I^P} \omega_J dx_{J1} \wedge \ldots \wedge dx_{Jq},$$

$$\mu \wedge \omega = \sum_{K\in I^{P+q}} \gamma K dx_{K1} \wedge \ldots \wedge dx_{K_{p+q}}$$

Corollary *The R-module Ω^χ is closed under χ–wedging.*

B. Criterion

When Solomon showed that Ω^G is generated over R by the df_i as an exterior algebra, he used the fact that the df_i wedge to Q vol. We follow ideas in Solomon's original proof to give a condition for n 1-forms to generate Ω^χ.

Proposition Let $\omega_1, \ldots, \omega_n$ be χ-invariant 1-forms. The forms $\omega_{I1} \lambda \ldots \lambda \omega_{Ip}$ for $I \in I$ p and $p \geq 0$, generate Ω^χ over R if and only if

$$\omega_1 \lambda \ldots\ldots\ldots\ldots \lambda \omega_n = Q_{\chi\det}{}^{vol}.$$

C. Criterion Satisfied

Now that we have a criterion for a set of 1-forms to generate Ω^χ, we wonder if there exist any forms that satisfy the criterion. In the case $\chi = \det^{-1}$, we have already constructed such forms. We built $(\det^{-1})$-invariant 1-forms, $\mu_1, \ldots\ldots\ldots\ldots, \mu_n$, whose exterior product is Q^{n-1} vol. We called the μ_i basic anti-invariant forms because they satisfy the criterion in Proposition:

$$\begin{aligned} \mu_1 \lambda \ldots \lambda \mu_n &= Q^{1-n} \mu_1 \wedge \ldots \wedge \mu_n \\ &= Q^{1-n} Q^{n-1} vol \\ &= 1 vol \\ &= Q_{\det^{-1}.\det} vol. \end{aligned}$$

The case $\chi = \det^{-1}$ is not just an example. We will use the basic anti-invariant forms to satisfy the criterion in Proposition for arbitrary χ.

LOGARITHMIC FORMS

We now discuss a few applications of the previous ideas to logarithmic forms. Some of these applications will appear in [12]. We have previously only considered differential forms with polynomial coefficients, but now consider differential forms with rational functions as coefficients. The S-module of logarithmic p-forms with poles along A (see also [13], p. 124) is defined as

$$\left\{ \frac{\omega}{Q_{\det^{-1}}} : \omega \in \Omega^p \, and \, \omega \wedge d\alpha_H \in \alpha_H \Omega^{p+1} \, for \text{ all } H \in A \right\}$$

G. Ziegler [16] extends this definition to multi-arrangements of hyperplane, hyperplane arrangements in which each hyperplane is given a positive integer multiplicity. We apply his definitions to our context of reflection groups and semi-invariants: Let A_χ

be the multi-arrangement consisting of each hyperplane $H \in A$ counted with multiplicity $\alpha_H(\chi)$, i.e., the multi-arrangement defined by Q_χ.

Corollary

$$\Omega^p(A) = \frac{1}{Q} S \otimes_R (\Omega^P)^{\det^{-1}}$$

Proof: If $\mu_{1,\ldots\ldots}$, μ_n are basic anti-invariant forms, then $\mu_1 \wedge \ldots \wedge \mu_n = Q^{n-1}$ vol. Hence by above propositions is freely generated by $Q^{-1}\mu_{I1} \wedge \ldots \wedge Q^{-1}\mu_{Ip}$, $I \in I^p$.

The above corollary is shown with a different argument in [13].

A. Basic χ- Forms

We now introduce a construction of differential 1-forms from derivations and polynomials that behave well with respect to semi-invariance. The idea is to replace each ∂x_i with dx_i and each x_i with ∂x_i, bar the complex scalars, and then apply the resulting operator to a polynomial.

We show here how above ideas are used to compute semi-invariants explicitly. Specifically, we show how to find all the semi-invariants for the exceptional irreducible reflection groups. We make some remarks about the irreducible reflection groups and display a reduced form of the Hilbert Series of χ-invariants for each of the exceptional groups. We indicate the basic χ-forms for the higher dimensional exceptional groups as well.

B. Irreducible Reflection Groups

We remarked above that every reflection group is either irreducible or the direct product of irreducible components, each of which is itself a reflection group. We thus focus on semi-invariants of the irreducible reflection groups. The irreducible reflection groups consist of three infinite families and thirty-four exceptional groups. Shephard and Todd gave each group a serial number: the infinite families are numbered 1 through 3 and the exceptional groups are numbered 4 through 37. The exceptional groups range in dimension from 2 to 8 and are labeled G_4 through G_{37}.

We have multiplied each series by $(1 - x^{d1}) \ldots (1 - x^{dn})$, where d_i is the degree of f_i, to obtain a polynomial called the reduced Hilbert series that ignores the contribution of the basic invariants. Corollary implies that the quotient is indeed a polynomial, and that this polynomial factors. The coefficient of x^iy^j is the dimension over R of the space of j-forms of homogeneous polynomial degree i. Nolan Wallach suggested factoring the polynomials with negative exponents, which prompted the idea to knock Stanley polynomials and basic derivations together. The Hilbert series were computed from character tables using a version of Molien's theorem and the software GAP and Mathematica.

C. Two-dimensional Groups

We now restrict our attention to the two-dimensional exceptional groups. We compute basic derivations for the duality groups with a simple formula: $\phi_i = df_{n-i} \Theta f_n$. The numerology of a duality group suggests this formula, but perhaps the formula in some sense also sheds light on the numerology. Each basic χ-form of a duality group is either $df_i Q_\chi$ or ϕ_i o Q_χ. With nonduality groups, these constructions may give zero for some i. We then substitute f_nf_i (for some i) for f_n.

Klein [13] explored the invariant theory for the projective tetrahedral, octahedral, and icosahedral groups in detail. There are three important invariant polynomials for each projective group, f, h, and t. Up to a scalar, the polynomial h is the Hessian of f and the polynomial t is the Jacobian of f and h. Here, f, h, and t for the tetrahedral, octahedral, and icosahedral groups explicitly. Any invariant of the corresponding reflection group can be written as a product of the f, h, and t. We give the basic invariants in terms of f, h, and t and give the basic derivations in terms of co-knocking f_n with df_i. We write each Stanley polynomial as a product of the f; h, and t. The Stanley polynomials were found by examining the effect of group generators using Mathematica. Using these theories, the reader can compute any semi-invariant form for a two-dimensional exceptional group from the appropriate polynomial f alone!

D. Higher Dimensional Groups and Tables

The basic χ-forms for the rest of the exceptional groups follow the same patterns. We have computed the basic derivations and basic χ-forms for the groups G_{25}, G_{26}, G_{28}, and G_{32}. The other higher dimensional groups carry only the trivial and $\det^{-1}$ multiplicative characters. Hence, their semi-invariants can be constructed from the basic derivations computed in [6] and [8]. We have also computed basic derivations and basic χ-forms for the non-duality group G_{31}. We indicate the results here, but forego the explicit calculations. As with the two-dimensional groups, basic derivations for the duality groups can be computed with the simple formula: $\phi_i = df_{n-i} \Theta f_n$. Again, each basic χ-form of a duality group is either $df_i Q_\chi$ or ϕ_i o Q_χ. The reduced Hilbert series and indicates how to choose $df_i Q_\chi$ or ϕ_i o Q_χ. Each factor $(1 + x^dy)$ in the reduced series corresponds to $df_i Q_\chi$, where d is the degree of df_i (exponent). Each factor $(1 + x^{-e}y)$ corresponds to ϕ_i o Q_χ, where e is the degree of ϕ_i (co-exponent). Again, with our non-duality group G_{31}, the construction $\phi_i = df_{n-i} \Theta f_n$ gives zero for one i. We then substitute $f^2{}_n$ for f_n.

CONCLUSION & FUTURE WORK

We provide a survey the research in the area of conceptions of mathematics to mathematics education research. Analogous results hold for vector fields, or derivations. Let Υ^χ be the module of χ-invariants in the exterior algebra of derivations. Because the group action differs here, these case where $I_1 \neq 1$ is the same as in the original lemma, and hence $Q\times$ also divides the exterior product of two elements in Υ^χ (the proof is analogous to the case of Ω^{χ}). The criterion for n derivations to generate Υ^{χ} via χ-wedging is also slightly different: they must χ-wedge to $Q_{\chi^{-\det -1}} \frac{\partial}{\partial \infty_n} \wedge ... \wedge \frac{\partial}{\partial \infty_n}$ instead of $Q_{\chi^{-\det}} dx_1 \wedge ... \wedge dx_n$. This follows from the fact $dx_1 \wedge .... \wedge dx_n$ is (det^{-1})-invariant while $\frac{\partial}{\partial \infty_1} \wedge .... \wedge \frac{\partial}{\partial \infty_n}$ is det-invariant. Finally, we note that the correspondence between di_erential p-forms (in Ω^{p}) and (n - p)-forms in Υ (the exterior algebra of derivations) induces a module isomorphism between Ω^{χ} and $\Upsilon^{\chi-\det}$.

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